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Fatigue crack growth in degrading elastic components of nonlinear structural systems under random loading

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Abstract

Based on fracture mechanics and theory of random processes, a probabilistic approach is proposed to analyze fatigue crack growth, fatigue life and reliability of degrading elastic structural components in nonlinear structural systems. Both the material resistance to fatigue crack growth and the time-history of stress are assumed to be random. The effect of slow degradation of structural stiffness due to fatigue crack growth is taken into account. Analytical expressions are obtained for the special case where the random stress is a narrow-band process. Numerical examples are given for two nonlinear structural systems important in practice assuming the randomized Paris–Erdogan type crack growth law is applicable. To validate the approach, the results are compared with those obtained from simulation. © 1999 Published by Elsevier Science Ltd. All rights reserved.

Keywords: Fracture mechanics; Random fatigue crack growth; Stochastic averaging; Fatigue life; Digital simulation; Random loading

1. Introduction

Fatigue is known to be a major cause of failure of a large number of structural components. From a fracture mechanics point of view, fatigue damage of a component subject to dynamic loading can be measured by the size of the dominant crack, and failure occurs when this crack reaches a critical size. It is widely recognized that fatigue crack growth is fundamentally a random phenomenon that can only be quantified in terms of probability and statistics. The two main reasons for the randomness in fatigue crack growth behavior are the random material resistance to fatigue crack growth and the random loading. During the last decade, several stochastic models have been proposed for the analysis of fatigue

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crack growth in elastic structural components with random material resistance under constant amplitude loading (Lin and Yang, 1985; Kozin and Bogdanoff, 1989; Ortiz and Kiremidjian, 1986). To account for the effect of random stress fluctuations, Sobczyk (1986) proposed the replacement of the stress-intensity-factor range in a deterministic crack growth law with an equivalent range, i.e., the root-mean-square value of the stress intensity factor range. Bolotin (1989) derived an asymptotic expression for estimating fatigue crack growth under random loading.

Recently, a probabilistic analysis of fatigue crack growth in elastic components with random material resistance to fatigue crack growth of linear structural systems under random loading has been presented by Zhu et al. (1992). It bases on fracture mechanics and the principle of Stratonovich's stochastic averaging (Stratonovich, 1963), and is applicable when fatigue growth is a slow process compared with the stress process, which is the case for high cycle fatigue. The random material resistance is modeled by multiplying a random process to a suitable deterministic crack growth equation. Closed form solutions for the first and second order probability densities of the stationary narrow-band random stress process are known and they can be used to determine the statistics of fatigue life of the components.

In the present paper, the above strategy is generalized to analyze fatigue crack growth in degrading elastic components with random material resistance to fatigue crack growth of nonlinear structural systems under stationary wide-band random loading. The method of stochastic averaging of the energy envelope (Zhu and Lei, 1988; Zhu and Lin, 1991), which is efficient for analyzing response of nonlinear systems under random loading, is employed to evaluate the probabilistic distribution of the random stress process. The slow degradation of structural stiffness due to crack growth is taken into account. Two important nonlinear structural systems are analyzed as numerical examples. In order to verify the proposed approach, all the results are compared with those obtained from digital simulation.

2. General analysis procedure of fatigue crack growth

Neglecting the secondary factors in a deterministic model of fatigue crack growth in opening mode, which is the predominate mode of macroscopic fatigue crack, the fatigue crack growth in elastic material under cyclic loading is governed by the following equation

$$\frac{\mathrm{d}a}{\mathrm{d}n} = f(a,\Delta s), \tag{1}$$

where a is crack size (half length for a though crack), n is the number of stress cycles, f is a non-negative function and Δs is stress range.

The two main reasons for the randomness in fatigue crack growth are random material resistance to fatigue crack growth and random stress in structural systems. Vikler et al. (1979) conducted a large replicate tests to identify the contribution of material inhomogeneity to the randomness observed in laboratory fatigue crack growth data. Two types of inhomogeneity behavior, which are called low frequency component and high frequency component, respectively, were noted. Both components have apparently random nature. To account for the random material resistance to fatigue crack growth, Eq. (1) may be randomized as follows

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \mu f(A,\Delta s,\underline{\mathrm{R}}), Y(\eta), \tag{2}$$

where $\underline{\mathrm{R}}$ is a vector of material-dependent parameter with random variables as its components describing the statistical scatter of material properties between specimens, and $Y(\eta)$ is a positive random process describing the random variability of material properties within a specimen. The statistical values,

$\underline{R}$ and $Y(\eta)$, are obtained from experimental data of fatigue crack growth under constant-amplitude cyclic loading. μ is the average number of cycles per unit time and the symbol for the crack size is capitalized to signify that it is now a random process. The stress range, Δs, remains deterministic.

The positive random process $Y(\eta)$ can be a random process of time, i.e. $Y(\eta)=Y(t)$. This random process model Y(t) has been investigated extensively (Lin and Yang, 1985; Sobczyk, 1986; Sobczyk and Spencer, 1992; Zhu et al., 1992). $Y(t)$ is used to account for the whole irregular variability including both high and low frequency components of fatigue crack growth, and $\underline{R}$ is taken to be a constant vector. Thus the two components of variability in crack growth curves observed by Vikler were mixed. But it is important and reasonable to split the two components in a stochastic model. So, in the present paper, $\underline{R}$ and $Y(t)$ are assumed to be a random vector and a stationary random process with $E[Y(t)]=1$, respectively, to account for the low and high frequency components of the material random variability. $Y(\eta)$ can also be a random function of the crack size $a(t)$, i.e., $Y(\eta)=Y(a)$ (Yang and Manning, 1990; Ortiz and Kiremidjian, 1986). Recently, the two stochastic models of fatigue crack growth under constant amplitude cyclic loading have been studied by Jiang and Zhu (1996). Random process $Y(\eta)$ can be either a function of time or a function of crack size. The models proposed by them yields better agreement with experimental data than other models available in literature.

Under stationary random loading, a logical generalization of Eq. (2) is to treat the stress range as the absolute value of the difference between a local maximum and a neighboring local minimum, i.e.,

$$\Delta S(t)=|S(t_1)-S(t_2)|,\ t_1\le t<t_2, \tag{3}$$

where t_1 and t_2 are the times at which two neighboring extrema of $S(t)$ occur. In this case, $\Delta S(t)$ becomes a stationary random sequence and independent of $Y(\eta)$, and μ is interpreted as the number of maxima per unit time. Then the modified version of Eq. (2) now reads

$$\frac{\mathrm{d}A}{\mathrm{d}t}=\mu f[A(t),\Delta S(t),\underline{R})]Y(\eta). \tag{4}$$

The fatigue crack size $A(t)$ is assumed to be a slowly varying random process compared with the stress process $\Delta S(t)$. This is a reasonable assumption for high cycle fatigue since the correlation time of $\Delta S(t)$ is expected to be much smaller than the fatigue life of a component. According to the Stratonovich–Khasminskii limit theorem (Stratonovich, 1963; Khasminskii, 1966), the crack size $A(t)$ is approximately a diffusive Markov process. Upon applying the stochastic averaging procedure (Khasminskii, 1966; Zhu, 1988, 1996), the Fokker–Planck equation for $A(t)$ is as follows

$$\frac{\partial p}{\partial t}=-\frac{\partial}{\partial a}[m(a)p]+\frac{1}{2}\frac{\partial^2}{\partial a^2}[\sigma^2(a)p], \tag{5}$$

where, when $Y(\eta)=Y(t)$,

$$m(a)=\mu E[f]E[Y]+\mu^2\int_{-\infty}^{\infty}\mathrm{cov}\left(\frac{\partial f}{\partial A}\bigg|_t,f_{t+\tau}\right)\mathrm{cov}[Y(t),Y(t+\tau)]\mathrm{d}\tau, \tag{6a}$$

$$\sigma^2(a)=\mu^2\int_{-\infty}^{\infty}\mathrm{cov}(f_t,f_{t+\tau})\mathrm{cov}[Y(t),Y(t+\tau)]\mathrm{d}\tau \tag{6b}$$

and, when $Y(\eta)=Y(a)$,

$$m(a) = \mu E[\bar{f}] + \mu^2 \int_{-\infty}^{\infty} \mathrm{cov}\left(\frac{\partial \bar{f}}{\partial A}\bigg|_t, \bar{f}_{t+\tau} \right) \mathrm{d}\tau \tag{6c}$$

and

$$\sigma^2(a) = \mu^2 \int_{-\infty}^{\infty} \mathrm{cov}\left(\bar{f}_t, \bar{f}_{t+\tau}\right) \mathrm{d}\tau, \tag{6d}$$

in which $p = p(a,t|a_0,t_0,\underline{r})$ is the conditional transition probability density of $A(t)$, $E[\cdot]$ denotes an ensemble average, and $\mathrm{cov}(\cdot,\cdot)$ denotes a covariance. Eq. (5) is solved subject to the initial condition

$$p(a,t|a_0,t_0,\underline{r}) = \delta(a - a_0),\ t = t_0. \tag{7}$$

By taking into account the low frequency component characterized by the random vector $\underline{R}$, the transition probability density function $p(a,t|a_0,t_0)$ is then determined according to the total probability theorem as

$$p(a,t|a_0,t_0) = \int p(a,t|a_0,t_0,\underline{r}) p(\underline{r}) \mathrm{d}\underline{r}. \tag{8}$$

The solution will provide a complete probability description of $A(t)$ or, more precisely, a Markovin approximation of $A(t)$. Obviously, $A(t)$ is a non-decreasing process. The reliability of the component at time t, conditional on a known initial crack size a_0 at time t_0 can be obtained from $p = p(a,t|a_0,t_0)$ as follows

$$R(a_{\mathrm{cr}},t|a_0,t_0) = \int_{a_0}^{a_{\mathrm{cr}}} p(u,t|a_0,t_0) \mathrm{d}u, \tag{9}$$

where a_{cr} is critical crack size. The conditional probability density of fatigue life T is then obtained as

$$p(T|t_0,a_0) = -\frac{\partial R}{\partial t}\bigg|_{t=T} \tag{10}$$

and the conditional mean and variance of fatigue life are

$$\frac{\partial p_i}{\partial t} = -\frac{\partial}{\partial a}[m_i(a)p_i] + \frac{1}{2}\frac{\partial^2}{\partial a^2}\left[\sigma_i^2(a)p_i\right] \tag{11a}$$

and

$$\sigma^2(T|a_0,t_0) = \int_0^{\infty} [T - E(T|a_0,t_0)]^2 p(T|a_0,t_0) \mathrm{d}T. \tag{11b}$$

In general, precise knowledge of the initial crack is not available, but its probability distribution can be assumed. Thus the unconditional counterparts of Eqs. (8)–(11a) and (11b) can be obtained by averaging over the range of initial condition.

Complications arise when the retardation (or acceleration) of the fatigue crack growth is considered. Dolinski and Colombi (1996) proposed that in that case stress extreme $S(t_2)$ in Eq. (3) should be modified as the effective minimum of the stress, and thus the stress range should be substituted by the effective stress range. Moreover, the fatigue crack growth involved load sequence effects due to the presence of parameters associated with overloads. In practical application the length of correlation of the sequence of extremes, N_{cor} = a dozen cycles or so, appears much shorter than the lengths of blocks consisting of retardation + post-retardation phase, N_B = several dozen of cycles, and the number of cycles to failure, N_F = several thousands cycles or more, is much longer than N_B's, i.e., $N_{cor} \ll N_B \ll N_F$. Under stochastic stationary loading, the retardation + post-retardation blocks may be considered to constitute a sequence of random, statistically independent, crack length increments having the same probability distributions. These properties ensure that the fatigue crack growth can still be reasonably approximated as a Markov process, and the above general analysis procedure can still be used. However, division of the process of fatigue crack growth into the blocks and cumbersome calculation efforts are involved in doing so. For simplicity, the retardation effect is neglected in the present paper. It has been shown that the predication of fatigue crack growth without considering the retardation effect is conservative (Jiang and Zhu, 1996).

In applying the above procedure, the mean and variance of $Y(\eta)$ are inferred from experimental data of fatigue growth under constant-amplitude cyclic loading, and a key ingredient is the covariance of $\Delta S(t)$ at two different times, which is needed to formulate the covariance of f_t and $f_{t+\tau}$, in Eqs. (6a) and (6b) or of $\bar{f}_t$ and $\bar{f}_{t+\tau}$ in Eqs. (6c) and (6d). Of course, the covariance of $\Delta S(t)$ can be obtained if the second order probability density of $\Delta S(t)$ is known. Unfortunately, to the authors' knowledge, this latter information is not available for a general stationary stress process. The special case of a stationary narrow band Gaussian stress process has been studied (Zhu et al., 1992). In what follows, we will investigate the case in which the stress in the elastic structural component with crack is proportional to the displacement response of the whole nonlinear structural system under wide-band random loading.

3. Nonlinear structural systems with cracked elastic component under wide-band random excitation

The displacement of a SDOF nonlinear structural system is governed by the following non-dimensional differential equation

$$X'' + 2\zeta_0 X' + g(X,X') = \zeta(\tau), \tag{12}$$

where x, ζ_0 and $g(x,x')$ are displacement, viscous damping ration and nonlinear restoring force of the system, respectively, and a prime expresses the derivative with respect to $\tau = \omega_0 t$, ω_0 is the frequency of the corresponding degenerated linear system, which is a constant value.

Considering degradation of the stiffness due to crack growth, Eq. (12) is then modified into

$$X'' + 2\zeta_0 X' + g(X,X',A) = \xi(\tau), \tag{13}$$

where $g(x,x',a)$ denotes the slowly degrading nonlinear restoring force of the system. If $\xi(\tau)$ is a stationary wide-band random process which can be modeled as a Gaussian white noise with intensity $2D$, the method of Stochastic averaging of energy envelope (Zhu and Lei, 1988; Zhu and Lin, 1991) leads to the following Fokker–Planck equation

$$\frac{\partial p}{\partial \tau} = -\frac{\partial}{\partial e}[U(e,a)p] + \frac{1}{2}\frac{\partial^2}{\partial e^2}[V^2(e,a)p], \tag{14}$$

where $p(e,\tau|e,\tau_0,a)$ is the conditional transition probability density of energy envelope defined as $e(\tau) = x'^2/2 + G(x,a)$ with initial condition $p(e,\tau|e,\tau_0,a) = \delta(e-e_0)$ when $\tau = \tau_0$, and

$$U(e,a) = -(2\zeta_0 \Phi(e,a) + A_t(e,a))/T(e,a) + D,$$

$$V^2(e,a) = \frac{2D\Phi(e,a)}{T(e,a)},$$

$$\Phi(e,a) = 2\int_{-\lambda}^{\lambda} \sqrt{2e - 2G(x,a)}\mathrm{d}x,$$

$$T(e,a) = 2\int_{-\lambda}^{\lambda} \frac{\mathrm{d}x}{\sqrt{2e - 2G(x,a)}}, \tag{15}$$

in which $A_r(e,a)$ is the energy dissipation by the non-viscous damping in one cycle, $G(x,a)$ is the potential energy of the nonlinear system when the current crack size is a, and λ is the solution of the equation $G(\lambda,a) = e$.

The conditional stationary probability density of $E(\tau)$ can be obtained from the solution of the reduced version of Eq. (14) without the time-derivative term

$$p(e|a) = C_1 T(e,a)\exp\left\{ -\int_0^e \left[\frac{2\zeta_0}{D} + \frac{A_r(e',a)}{D\Phi(e',a)} \right] \mathrm{d}e' \right\}, \tag{16}$$

where C_1 is a normalizing constant.

The conditional transition probability density function $p(e_2,\tau'|e_1,a)$ of $E(t)$ can be evaluated by numerical solution of Eq. (14). According to the characteristics of the Markov process $E(t)$, the second order probability density of the stationary process $E(t)$ is

$$p(e_1,e_2;\tau'|a) = p(e_2,\tau'|e_1,a)p(e_1|a), \tag{17}$$

where $\tau' = \tau_2 - \tau_1$. Defining an amplitude envelope process $P(t)$ by $G[P(\tau),a] = E(\tau)$, the conditional first and second order probability densities of the stationary process $P(t)$ are then

$$p(\rho|a) = p(e|a)\frac{\mathrm{d}G(\rho,a)}{\mathrm{d}\rho}\bigg|_{e=G(\rho,a)}, \tag{18}$$

$$p(\rho_1\rho_2;\tau'|a) = p(e_1,e_2;\tau'|a)\frac{\mathrm{d}G(\rho_1,a)}{\mathrm{d}\rho_1}\frac{\mathrm{d}G(\rho_2,a)}{\mathrm{d}\rho_2}\bigg|_{e_k=G(\rho_k,a)}, \tag{19}$$

in which $k = 1,2$

If the displacement response of nonlinear structural system is a rather narrow band process, the displacement range can be replaced approximately by $2P(t)$. Furthermore, assume that the stress in the elastic component where crack propagates is proportional to the displacement of the whole nonlinear system. Then $\Delta S(t)$ is also proportional to $2P(t)$. The covariance in Eqs. (6a–d) can be evaluated in terms of Eqs. (18) and (19). Thus, the general procedure described in the last section can be applied by combination with the statistics derived in this section.

The average number of cycles per unit time μ can be evaluated by

$$\mu(a) = \int_{-\infty}^{\infty} x'p(0,x'|a)\mathrm{d}x', \tag{20}$$

where

$$p(x,x'|a) = \left.\frac{p(e|a)}{T(e,a)}\right|_{e=x'/2+G(x,a)}. \tag{21}$$

In numerical calculations, the entire range of the crack size is divided into m small segments; namely, $[a_0,a_1]$, $[a_1,a_2]$,..., $[a_{m-1},a_m]$. Since the degradation of stiffness of the structural system is a slowly varying process, in each small crack size segment, the stress range and the average number of cycles per unit time can be assumed as being a random process and value independent of the crack size, respectively. The corresponding Fokker–Planck equation for $A(t)$ in the i-th crack size region $[a_{i-1},a_i]$ $(i = 1,2,..., m)$ is then obtained as

$$\frac{\partial p_i}{\partial \tau} = -\frac{\partial}{\partial a}[m_i(a)p_i] + \frac{1}{2}\frac{\partial^2}{\partial a^2}\left[\sigma_i^2(a)p_i\right], \tag{22}$$

with the initial condition

$$p_i(a,\Delta\tau_i|a_{i-1},\underline{\mathrm{r}}) = \delta(a-a_{i-1}), \text{ when } \Delta\tau_i = 0, \tag{23}$$

where $p_i(a,\Delta\tau_i|a_{i-1},\underline{\mathrm{r}})$ $a_{i-1} \le a \le a_i$ is the conditional transition probability density function of $A(\tau)$ which takes the values between a_{i-1} and a_i, and $m_i(a),\sigma_i^2(a)$ can be evaluated according to Eqs. (6a) and (6b) or to Eqs. (6c) and (6d), respectively. In the i-th crack size segment, the average number of cycles per unit time μ_i can be approximately evaluated by Eq. (20) as

$$\mu_i = \mu\left(\frac{a_i + a_{i-1}}{2}\right), \tag{24}$$

and the first and second order conditional probability densities of the stress process in this crack size segment can also be approximately treated to be independent of crack size and take the values as

$$p_i(\rho) = p\left(\rho\left|\frac{a_i + a_{i-1}}{2}\right.\right), \tag{25}$$

$$p_i(\rho_1,\rho_2;\tau') = p\left(\rho_1,\rho_2;\tau'\left|\frac{a_i + a_{i-1}}{2}\right.\right). \tag{26}$$

Finally, due to the characteristics of the Markov process $A(t)$, the conditional transition probability density function $p(a,\tau|a_0)$ of $A(t)$ which takes the value in the range $[a_{i-1},a_i]$ is

$$p(a,\tau|a_0,\underline{\mathrm{r}}) = \underbrace{\int_{a_{i-2}}^{a_{i-1}}\int_{a_{i-3}}^{a_{i-2}}\cdots\int_{a_0}^{a_1}}_{i-1} p_i(a,\Delta\tau_i|y_{i-1},\underline{\mathrm{r}})p_{i-1}(y_{i-1},\Delta\tau_{i-1}|y_{i-2},\underline{\mathrm{r}})\ldots p_1(y_1,\Delta\tau_1|a_0,\underline{\mathrm{r}})\mathrm{d}y_{i-1}\,\mathrm{d}y_{i-2}\ldots\mathrm{d}y_1, \tag{27}$$

where

$$\sum_{k=1}^{i}\Delta\tau_k = \tau.$$

In discretion of the entire range of the crack size, mesh size should be selected appropriately. A smaller mesh size results in more accurate statistical results in each segment but produces larger

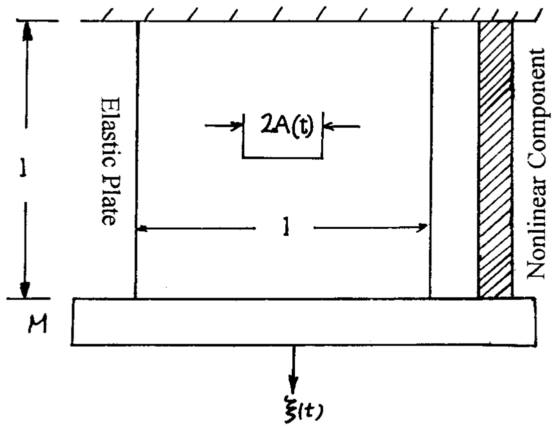

Fig. 1. Nonlinear system with crack growth in linear component under random loading.

accumulative error. We successively divide the crack size segment into halves until the numerical results of the conditional transition probabilities at the corresponding discretized points estimated from the two adjacent discretion converge with the admissible error.

4. Example

Consider a thin square plate, $l \times l$, with an initial central crack of length $2a_0$ and a structural component in parallel supporting an infinitely rigid heavy mass M at their end (see Fig. 1). The mass M is subjected to a stationary wide-band load process $\xi(t)$ perpendicular to the crack with a one side power spectral density G_0. The plate is idealized to be elastic, massless, homogeneous, isotropic and with light linear damping, whereas plasticity occurs in the other structural component and the total resorting force of the system is nonlinear under severe loading. Degradation of plate stiffness due to crack growth is considered. The analysis procedure proposed in the present paper is applicable, as long as the degradation is a slow process.

The displacement $X(t)$ satisfies the differential equation

$$M\ddot{X}(t) + C\dot{X}(t) + K\{\theta[A(t)]X + g_2(X,\dot{X})\} = \xi(t), \tag{28}$$

where C denotes system damping and is considered to be time-invariant, K is the stiffness of the plate, and $K{\cdot}g_2(x)$ denotes the nonlinear stiffness contributed by the nonlinear component. The function $\theta(a)$ accounts for the degradation of plate stiffness due to crack growth and is approximated by the following polynomial expression (Grigoriu, 1990)

$$\theta(a) = 1 - 1.708u^2 + 3.081u^4 - 7.036u^6 + 8.928u^8 - 4.266u^{10}, \tag{29}$$

where $u = 2a/l$.

The following Paris–Erdogan law for crack growth is considered:

$$\frac{\mathrm{d}A}{\mathrm{d}\tau} = \mu\delta(\Delta k)^{\upsilon}, \tag{30}$$

where δ and υ are two constant material parameters, μ is the average number of cycle per unit time, which now depends on the crack size due to stiffness degradation, $\Delta k(t)$ denotes the range of the stress intensity factor, and τ is nondimensional time $\tau=\omega_0 t$ with $\omega_0=K/M$.

Let $h(a)$ be the stress intensity factor at the crack tip corresponding to a crack length $2a$ and unit stress range. An approximate expression for $h(a)$ is (Grigoriu, 1990):

$$h(a)=\sqrt{u}(0.467-0.514u+0.960u^2-1.421u^3+0.782u^4). \tag{31}$$

In this example, the stress in the plate where crack propagates is proportional to the displacement of the nonlinear system. Assuming the displacement response of nonlinear structural system is a rather narrow band process, the displacement range can be replaced approximately by $2P(t)$. Thus $\Delta k=K\theta(a)\Delta\rho h(a)=2K\theta(a)\rho h(a)$ and Eq. (30) can be rewritten as

$$\frac{\mathrm{d}A}{\mathrm{d}\tau}=\mu\lambda Q(A)P^{\upsilon}(\tau), \tag{32}$$

where

$$\lambda=2^{\upsilon}\delta K^{\upsilon},\quad Q(A)=h^{\upsilon}(A)\theta^{\upsilon}(A). \tag{33}$$

Eq. (32) is a special case of Eq. (4) with ΔS replaced by $2S$ and $Y(\eta)=1$. However, the response of the structural system, e.g., $P(t)$ and μ now depend on the current crack size due to the plate stiffness degradation. Under the assumption that stress in the plate is equal to displacement multiplied by plate stiffness, plate stiffness degradation has a direct effect in reducing the stress in the plate, although it can also increase the displacement response. Therefore, the stress in the plate is reduced compared with the case without plate stiffness degradation. Moreover, stiffness degradation also reduces the average number of cycle per unit time. For the special case of a linear system, analytical expressions are given which clearly indicate that crack growth rate is reduced due to plate stiffness degradation (Zhu et al., 1992; Grigoriu, 1990). For the nonlinear system considered in this paper, the increment of displacement response due to plate stiffness degradation is smaller compared with that of the corresponding degenerated linear system because of the existence of the nonlinear responding force, but plate stiffness degradation still has a direct effect in reducing stress in the plate, and the average number of cycle per unit time is also reduced. Therefore, plate stiffness degradation has the effect in prolonging the fatigue life of the structural system and this effect is more distinct when the material parameter υ becomes larger.

Applying the above analysis procedure, the following conditional transition probability density of $A(t)$ in the i-th crack size segment is obtained

$$p_i(a,\tau'|a_{i-1})=\frac{\exp\left\{-\left[b_i(a)-m_i\tau'\right]^2/2\sigma_i^2\tau'\right\}}{\sqrt{2\pi\tau'}Q(a)\Phi\left[m_i\sqrt{\tau'}/\sigma_i\right]}, \tag{34}$$

where $a_i\le a\le a_{i-1}$, $\tau'=\tau_2-\tau_1$, $\Phi(\cdot)$ is the standard normal distribution function, and

$$b_i(a)=\int_{a_{i-1}}^{a}\frac{\mathrm{d}u}{Q(u)},\quad m_i=\mu_i\lambda\int_0^{\infty}\rho^{\upsilon}p_i(\rho)\mathrm{d}\rho, \tag{35a}$$

$$\sigma_i^2=2\mu_i^2\lambda^2\int_0^{\infty}\left[\int_0^{\infty}\int_0^{\infty}\rho_1^{\upsilon}\rho_2^{\upsilon}p_i(\rho_1,\rho_2;\tau)\mathrm{d}\rho_1\,\mathrm{d}\rho_2-m_i^2\right]\mathrm{d}\tau, \tag{35b}$$

in which μ_i, $p_i(\rho)$ and $p_i(\rho_1,\rho_2;\tau)$ are evaluated according to Eqs. (24)–(26), respectively.

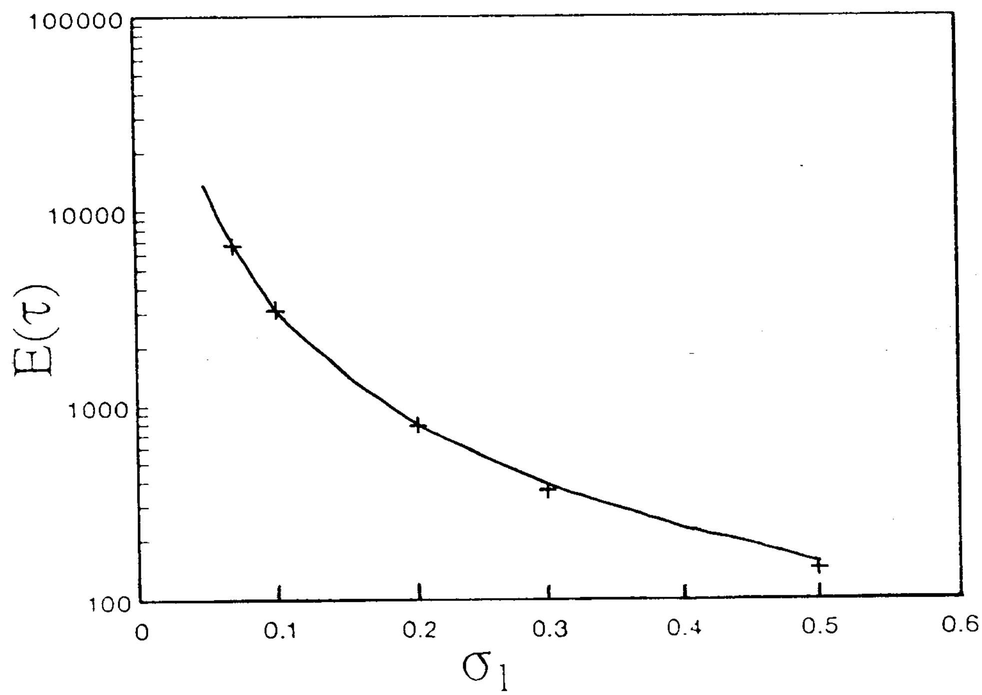

Fig. 2. Mean fatigue life vs load spectral in Duffing-type nonlinear system; ——— approximate result; + simulation result.

4.1. Duffing-type nonlinear structural system

The restoring force of the structural system is now in the form: $K[\theta(a)x+\gamma x^3]$; then

$$G(x,a) = K\left[\frac{1}{2}\theta(a)x^2 + \frac{1}{4}\gamma x^4\right]; \tag{36}$$

$$A_r(a) = 0. \tag{37}$$

The first order probabilistic density of $E(t)$ can be obtained from Eq. (16) as

$$p(e|a) = C_1 T(e|a)\exp\left[-\frac{4\zeta_0}{\pi G_0}e\right] \tag{38}$$

and the first order probabilistic density of $P(t)$ is

$$p(\rho|a) = p(e)|_{e=K[\theta(a)\rho^2/2+\gamma\rho^4/4]}K[\theta(a)\rho + \gamma\rho^3]. \tag{39}$$

The transition probability density of $E(\tau)$ is numerically evaluated from Eq. (14) by using the Crank–Nicolson implicit difference method. The second order probability density of $E(\tau)$ can be evaluated according to Eq. (17) and the second order probability density of $P(\tau)$ is then obtained from Eq. (19).

Numerical calculation is made with the following value parameters: $l = 0.254$ m, $M = 5.35$ kg,

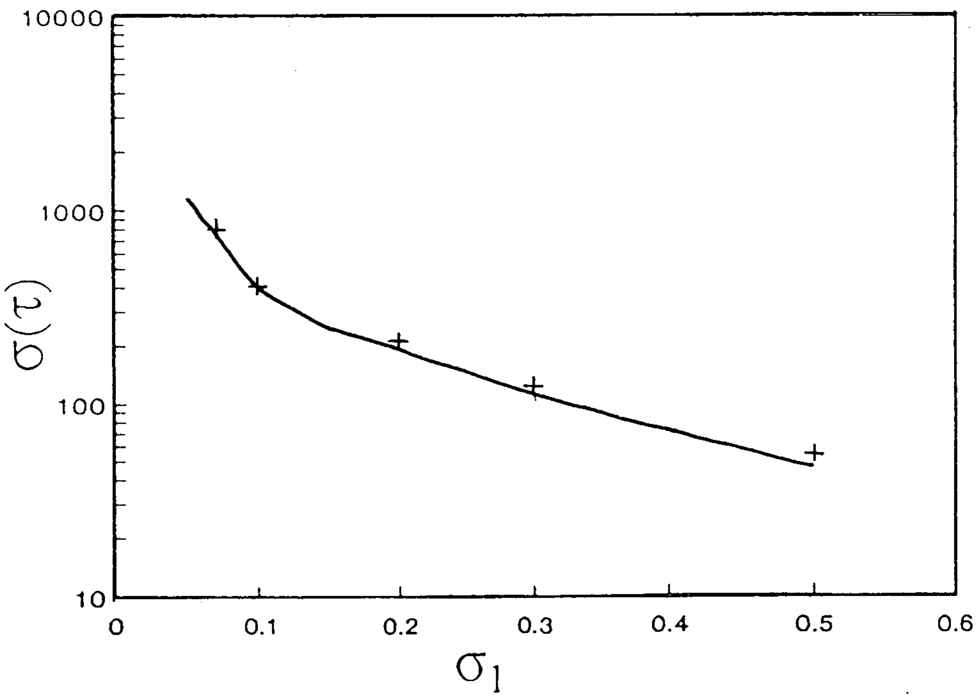

Fig. 3. Standard deviation of fatigue life vs load spectral level in Duffing-type nonlinear system; ——— approximate result; + simulation result.

$C = 4.375\ \text{kg s}^{-1}$, $K = 2.68 \times 10^3\ \text{N/m}$, $\delta = 6.6 \times 10^{-7}$, $\upsilon = 2.25$, $2a_0 = 0.00254\ \text{m}$, $2a_{\text{cr}} = 0.0254\ \text{m}$ and the thickness of the plate is 0.0254 m.

The mean and standard deviation of fatigue life in $\tau = \omega_0 t$, $\omega_0^2 = K/M$ for $\gamma = 0.2$ and a range of excitation intensity are shown in Figs. 2 and 3, where $\sigma_1 = \sqrt{G_0/M^2\omega_0^3}$. The probability density function of fatigue life and reliability function are shown in Figs. 4 and 5, respectively, with $\sigma_1 = 0.3$. It is seen from comparison of the approximate results (solid lines) with those from digital simulation (+) that the two results agree quite well. It is also verified that the present approach still yields good results even for strong nonlinearity, $\gamma = 1.0$, as long as the damping is small. However, when ζ_0 becomes large ($\zeta_0 > 0.2$), there is certain discrepancy between the approximate results and those from the simulation, and the theoretical results are on the safe side. The error is due to the fact that the displacement response of the nonlinear system in this case is not narrow-banded, as shown in Fig. 6. Wrischig and Light (1982) proposed an empirical prediction for the fatigue damage under a non-narrow band stress process by letting $D = \lambda D_{\text{N}}$, $\lambda < 1.0$, where D_{N} is the damage indicator under a narrow-band stress process. Therefore, the above observation agrees with Wrischig's conclusion.

Fig. 7 shows the effect of nonlinear term on the mean value of fatigue life. When excitation is small ($\sigma_1 < 0.08$), the nonlinear effect is insignificant. When excitation becomes larger, the fatigue life is prolonged due to the nonlinear term. This effect is more apparent when the material parameter υ becomes larger. The kurtosis value, defined as $K = E(X^4)/\sigma_x^4$ for a Duffing-type nonlinear system, can be shown to be smaller than 3. Therefore, the result agrees with Lutes' conclusion (Lutes et al., 1984) about the effect of non-normality on fatigue life.

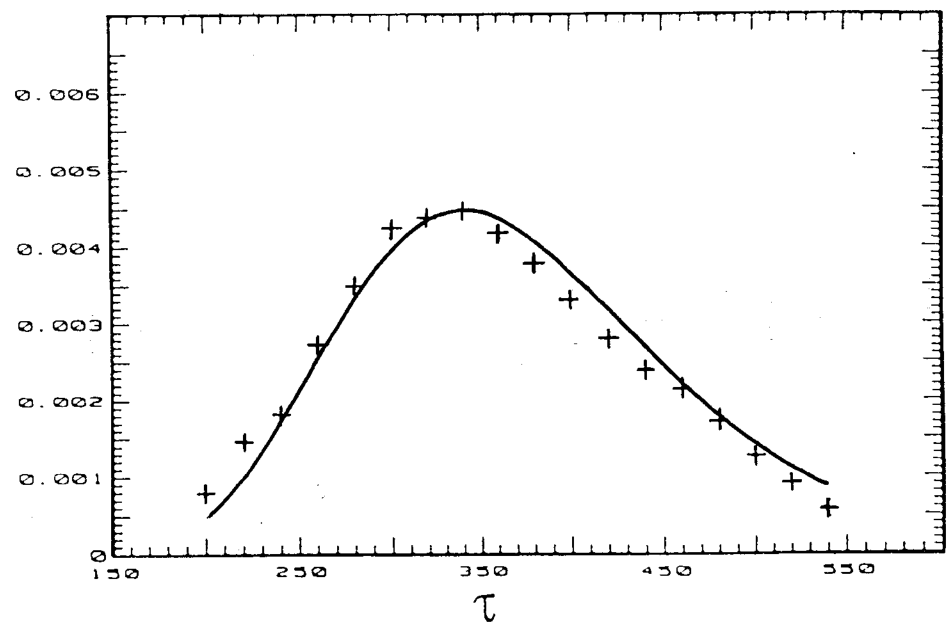

Fig. 4. Probability density function of fatigue life for $\sigma_1 = 0.3$ in Duffing-type nonlinear system; ——— approximate result; + simulation result.

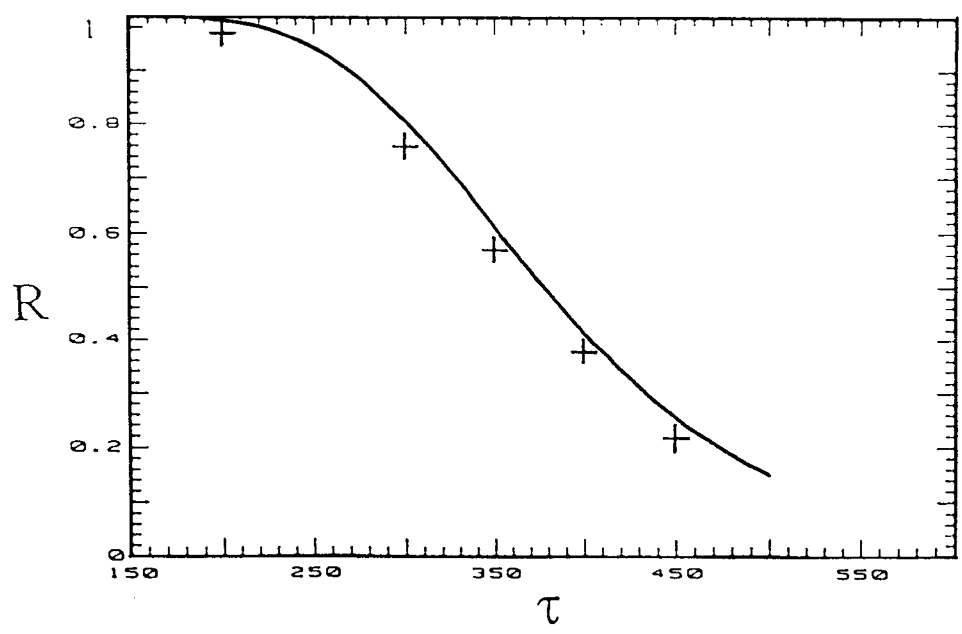

Fig. 5. Reliability function of fatigue life for $\sigma_1 = 0.3$ in Duffing-type nonlinear system; ——— approximate result; + simulation result.

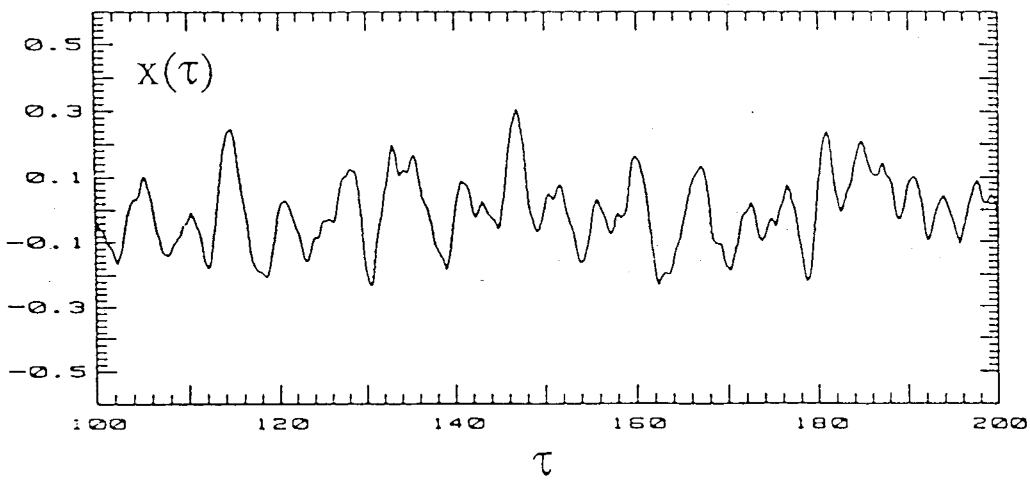

Fig. 6. Displacement response sample of Duffing-type nonlinear system when $\zeta_0 = 0.5$.

4.2. Hysteretic structural system

A hysteretic structural system, which is important in practice, is analyzed. Displacement $X(t)$ satisfies the following nonlinear differential equation

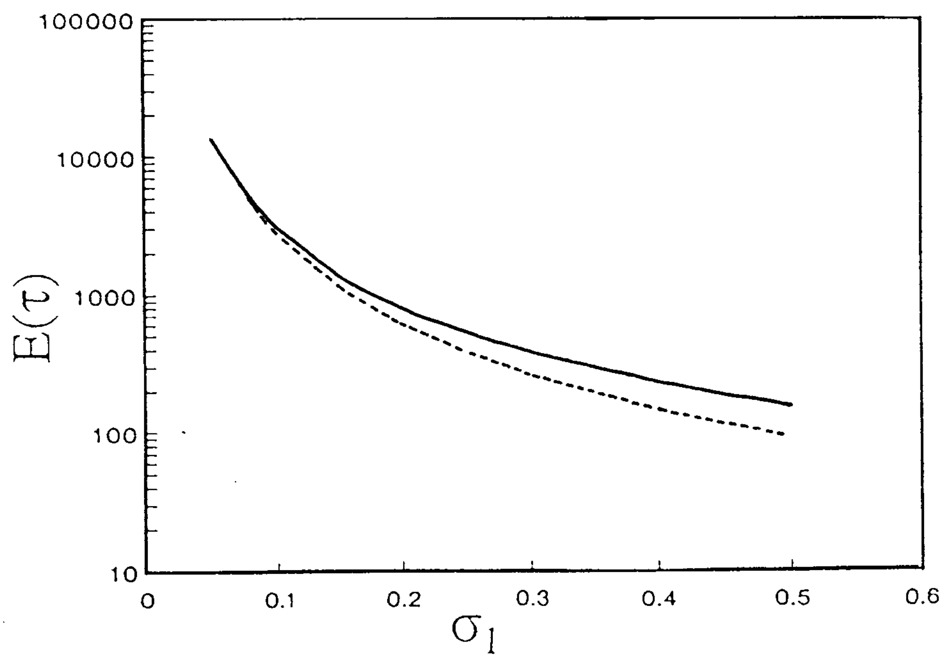

Fig. 7. Comparison of mean fatigue life for Duffing-type nonlinear system (————) and the corresponding degenerated linear system (– – – –).

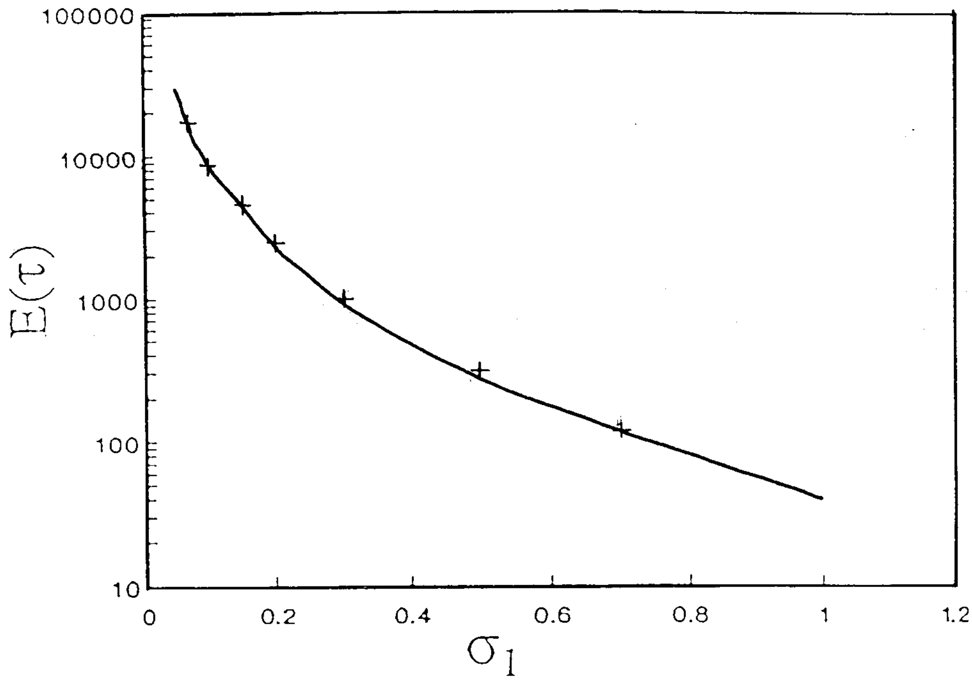

Fig. 8. Mean fatigue life vs load spectral level for $\alpha=0.5$ in hysteretic system; ——— approximate result; + simulation result.

$$M\ddot{X}(t)+C\dot{X}(t)+K\{\alpha\theta[A(t)]X(t)+(1-\alpha)Z(t)\}=\xi(t), \tag{40}$$

where Z is the hysteretic component of the resorting force, which can be described by the following first order nonlinear differential equation (Zhu and Lin, 1991)

$$\dot{Z}=-\gamma|X|Z|Z|^{n-1}-\beta|\dot{X}||Z|^{n}+A\dot{X}, \tag{41}$$

in which A, n, β and γ are positive constants controlling the hysteretic loop.

For the case of $A=n=1$, $\beta=\gamma=0.5$ and $\dot{x}>0$,

$$G(x,a)=K\left[\theta(a)\cdot\alpha x^{2}/2+(1-\alpha)(x+x_{0})^{2}/2\right] \text{ when } -\lambda\le x\le -x_{0}, \tag{42a}$$

$$G(x,a)=K\{\theta(a)\cdot\alpha x^{2}/2+(1-\alpha)[1-e^{-(x+x_{0})^{2}}]/2\} \text{ when } -x_{0}\le x\le\lambda, \tag{42b}$$

$$A_{r}(e,a)=K(1-\alpha)\left[4x_{0}-(x-x_{0})^{2}\right], \tag{43}$$

where λ and x_0 are determined by nonlinear equations which have been derived by Cai and Lin (1990). The expressions of $G(x,a)$ and A_r in the region and those for other values of A, n, β and γ are similar.

If $\xi(t)$ can still be modeled as a Gaussian white noise with intensity πG_0, the first and second order probability densities of the amplitude envelope process can be evaluated according to Eqs. (18) and (19), The statistics of fatigue life can be derived using Eqs. (9)–(11a) and (11b).

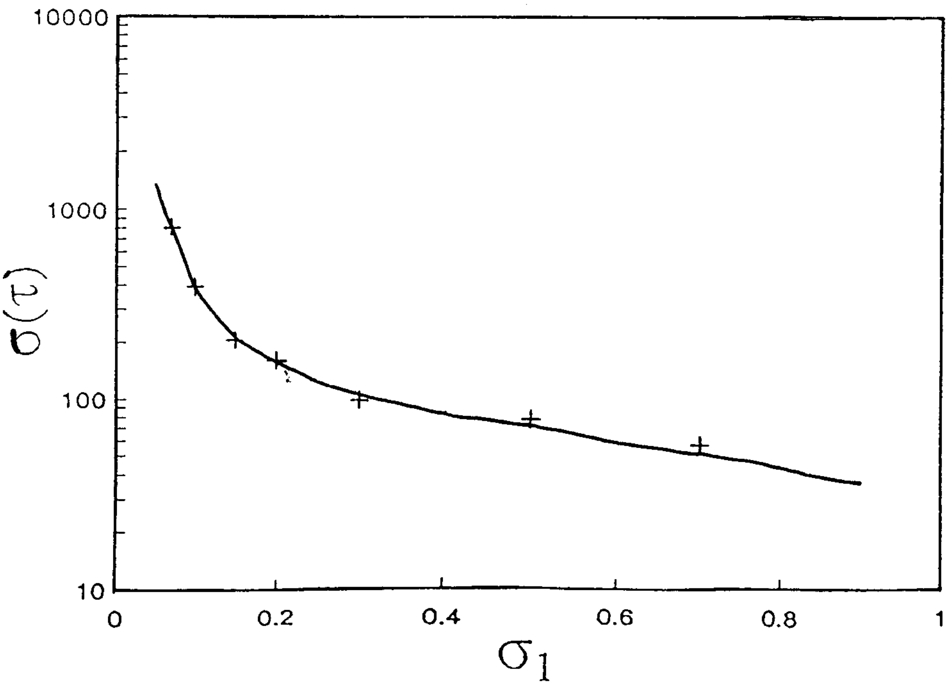

Fig. 9. Standard deviation of fatigue life vs load spectral level for $\alpha = 0.5$ in hysteretic system; ——— approximate result; + simulation result.

Numerical calculation is made with the same material parameter values as for the Duffing-type nonlinear system. The mean and standard deviation of the fatigue life in $\tau = \omega_0 t$ for $\alpha = 0.5$, $\alpha = 0.1$ and a range of excitation intensity are shown in Figs. 8–11, with $\sigma_1 = \sqrt{G_0/M^2\omega_0^3}$. For moderate hysteresis, $\alpha = 0.5$, it is seen that the theoretical results (solid lines) agree very well with those from simulation (+). However, For strong hysteresis, $\alpha = 0.1$, there is a certain discrepancy when excitation is in the intermediate level ($0.1 < \sigma_1 < 1.0$), and the theoretical results are on the safe side. The error is due to the fact that the displacement response for strong hysteresis is not narrow-band for intermediate level excitation, as shown in Fig. 12. The power spectral density of the displacement response in this case was shown to be broad-band (Iwan and Lutes, 1967). The above observation agrees with empirical results by Wrischig and Light (1982).

Similar conclusions can be drawn about comparisons of the reliability function, probability density function of fatigue life and the statistics of crack size.

Fig. 13 shows the effect of hysteresis on the mean value of fatigue life. When excitation is small ($\sigma_1 < 0.05$), the effect of hysteresis is not significant. In the interval of intermediate excitation strength, fatigue life is prolonged due to hysteresis compared with that in the linear system. This effect is more apparent when the material parameter υ becomes larger. For hysteretic system under an intermediate level of excitation, the kurtosis value $K < 3$. Therefore, it also agrees with Lutes' conclusion about the effect of non-normality on fatigue life. Moreover, it is noted that the effect of nonnormality of the stress process due to the hysteretic system on fatigue life is more significant than that of non-narrow band stress process.

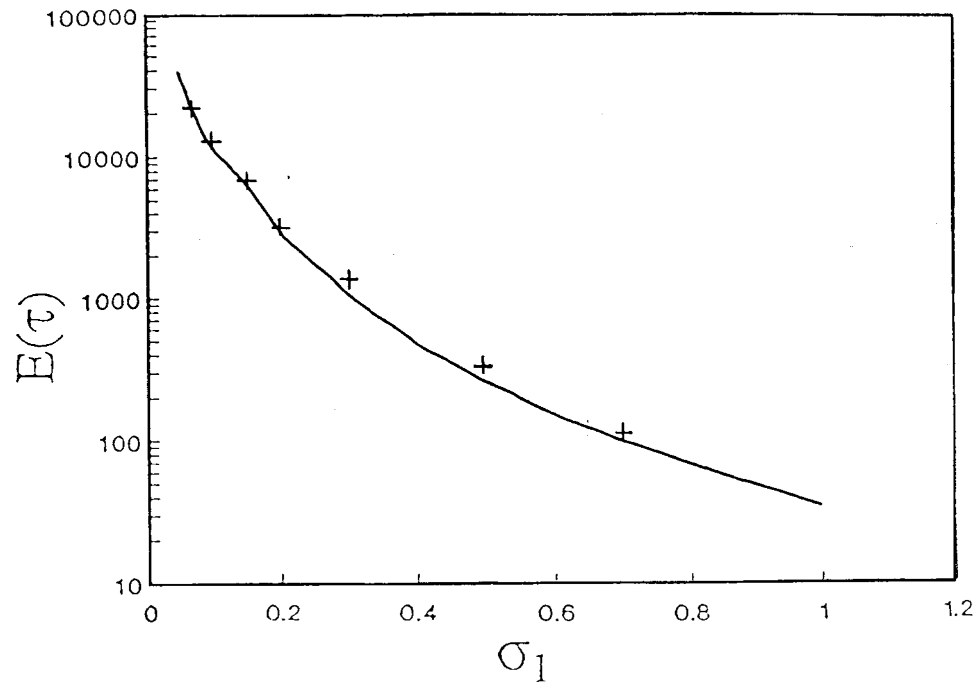

Fig. 10. Mean fatigue life vs load spectral level for $\alpha = 0.1$ in hysteretic system; ——— approximate result; + simulation result.

5. Conclusions

A probabilistic analysis of fatigue crack growth, fatigue life and reliability of degrading elastic structural components with random material resistance in nonlinear structural systems under random loading has been presented. It is based on fracture mechanics and the principle of Stratonvich's stochastic averaging, and is applicable when fatigue growth is a slow process compared with the stress process, which is the case for high cycle fatigue. The analysis has accounted for the coupling between response of nonlinear dynamic system and current state of crack size due to the degradation of structural stiffness. Since the degradation is a slow monatomic process, fatigue crack size is discretized into segments over an admissible range and, eventually, combined under the Markovian assumption to provided life time estimates. In particular, analytical expressions are given for the case that the random stress process is narrow-banded. Assuming the stress in the elastic component where crack propagates is proportional to the displacement of the whole nonlinear system, numerical results can be derived for the probabilistic distribution of the stress process using the method of stochastic averaging of the energy envelope. Numerical examples are given for two important nonlinear structural systems in practice where randomized Paris–Erdogan crack growth law is applicable. The effect of nonlinearty on fatigue life has also been indicated. Comparison of the approximate results with those obtained from simulation shows that the approach yields good results for wide ranges of parameter values.

As a first step of the analysis of fatigue crack growth of elastic components in nonlinear structural system, some simplifications and assumptions have been made in the present paper to simplify the analysis and computational efforts. Approximate results have been compared only with those from a digital simulation to validate the mathematical strategy. Comparison with experiments has not been made due to lack of experimental data by others and due to the limitation on performing experiments

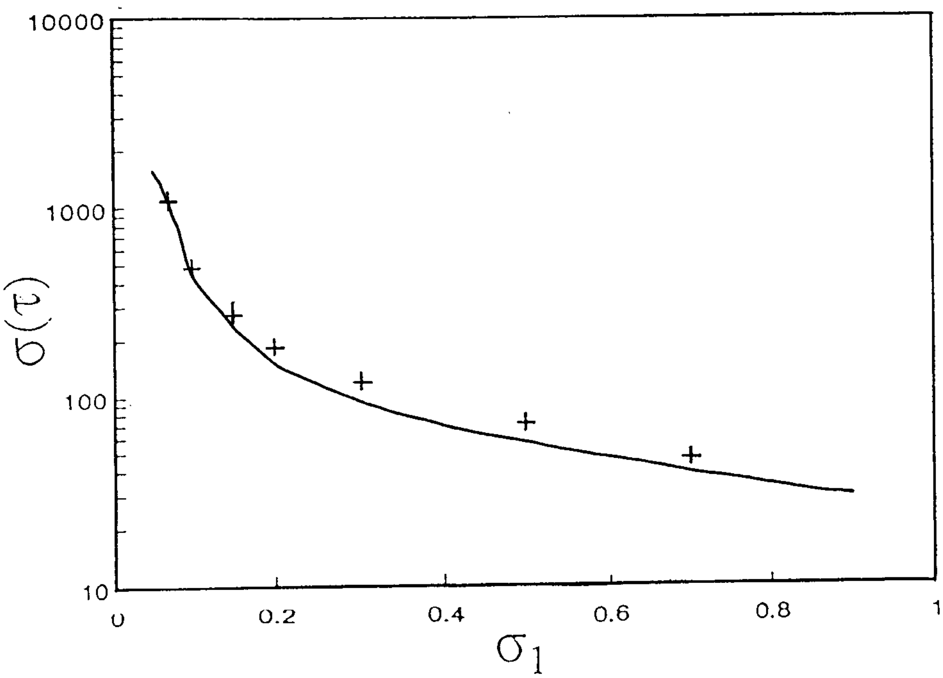

Fig. 11. Standard deviation of fatigue life vs load spectral level for $\alpha = 0.1$ in hysteretic system; ——— approximate result; + simulation result.

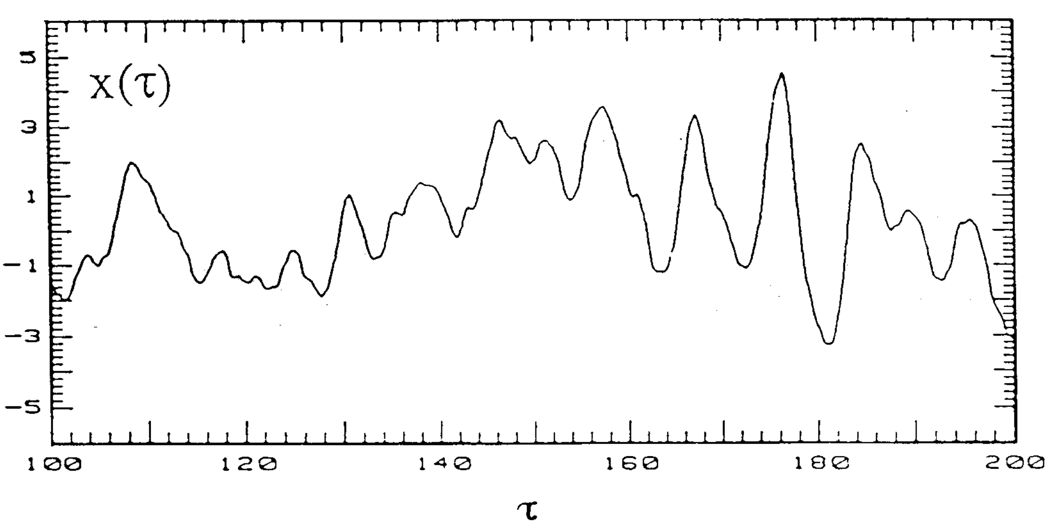

Fig. 12. Displacement response sample of hysteretic system for $\alpha = 0.1$ and $\sigma_1 = 0.5$.

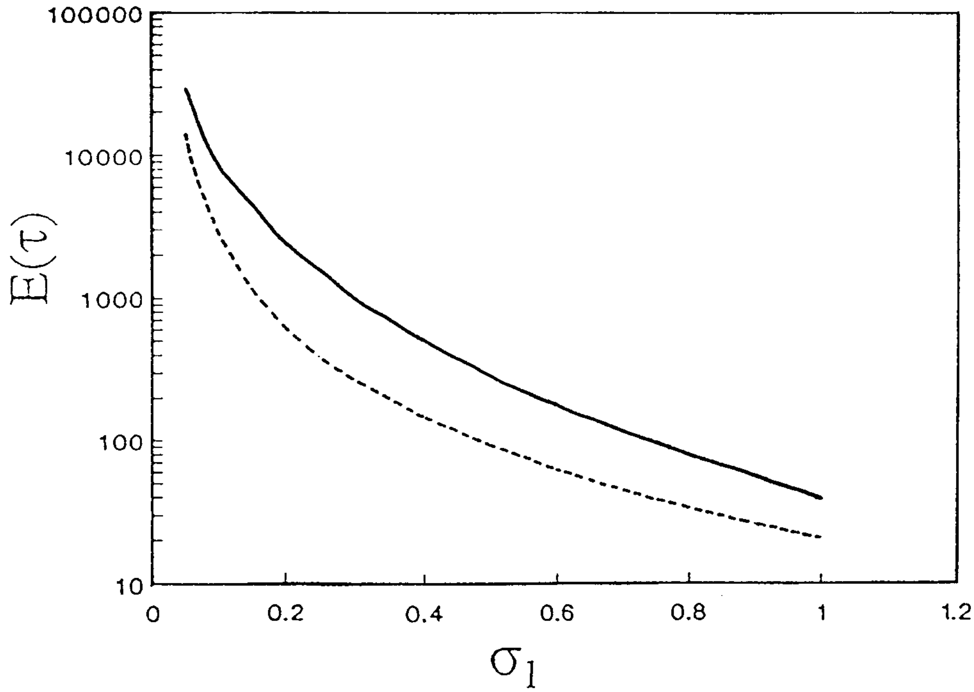

Fig. 13. Comparison of mean fatigue life for hysteretic system with $\alpha = 0.5$ (————) and the corresponding degenerated linear system (– – – –).

by the authors. More sophisticated analysis, which will involve retardation effect, statistical data of random material resistance to fatigue crack growth, and comparison with real experiments should be considered after more experience is gained through the present study.

Acknowledgements

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